

BETA-EXPANSIONS OF p -ADIC NUMBERS

KLAUS SCHEICHER¹, VÍCTOR F. SIRVENT², AND PAUL SURER³

ABSTRACT. In the present article, we introduce beta-expansions in the ring \mathbb{Z}_p of p -adic integers. We characterise the sets of numbers with eventually periodic and finite expansions.

1. INTRODUCTION

For a real number $\beta > 1$, the *beta-transformation* $T = T_\beta$ is defined for $x \in [0, 1]$ by

$$(1.1) \quad T(x) = \beta x - \lfloor \beta x \rfloor.$$

Denote $T^0(x) := x$ and $T^n(x) := T(T^{n-1}(x))$. By iterating this map, we obtain an expansion

$$x = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \cdots,$$

where $x_n = \lfloor \beta T^{n-1}(x) \rfloor$. We will call the sequence

$$(1.2) \quad d_\beta(x) := \bullet x_1 x_2 \cdots$$

the *beta-expansion* of x . This setting was introduced by Rényi and Parry (*cf.* [17, 18]). For beta-expansions of real numbers, there exist several results for the case when the base β is a Pisot number. Bertrand and Schmidt [8, 22] proved that if β is Pisot, then the set of eventually periodic beta-expansions consists of the non-negative elements of $\mathbb{Q}(\beta)$. Schmidt proved a partial converse of this statement, namely, if the non-negative elements of \mathbb{Q} have eventually periodic expansions, then β is a Pisot or Salem number.

In the case of Pisot numbers, it is proved that the associated shift space is sofic (*cf.* [8, 22]). Soficness could also be proved for Salem numbers of degree four (*cf.* [9]).

For real beta-expansions there is a long standing conjecture stated by Schmidt [22], that the set of eventually periodic beta-expansions corresponds to the rational numbers in the interval $[0, 1)$ if and only if β is a Pisot or Salem number. It has been proved in [9] that the beta-expansion of 1 is eventually periodic for Salem numbers of degree four. It is conjectured by Boyd [10], that this is also true for Salem numbers of degree six, but not for higher degrees.

Following Frougny and Solomyak [12], we say that an algebraic number $\beta > 1$ satisfies the finiteness property (F) if for each $x \in \mathbb{Z}[\beta^{-1}] \cap [0, 1)$ there exists a positive integer

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n such that $T^n(x) = 0$. Property (F) can only hold for Pisot numbers whose associated shift space is of finite type. The quadratic Pisot numbers that satisfy (F) are completely described in [12], but from the cubic case on a full characterisation seems to be very hard. Partial results, conditions and algorithms can be found, among other references, in [1, 2, 3, 6, 12, 23]. Moreover, beta-expansions have been studied in the context of formal Laurent series over finite fields. In this context the conjectures of Schmidt and Boyd have been proved (*cf.* [13, 19]).

In the present article, we deal with beta-expansions in the ring of the p -adic integers. We characterise the set of numbers with eventually periodic and finite expansions. In particular in Theorem 4.1, we prove that for β a Pisot-Chabauty number, the set of eventually periodic beta-expansions is $\mathbb{Q}(\beta) \cap \mathbb{Z}_p$. This is the equivalent of the result of Bertrand and Schmidt in the context of p -adic numbers. Furthermore, in Theorem 4.4, we prove an equivalent result to Schmidt's partial converse. In Theorem 5.4, we characterise the set of finite beta-expansions for a family of Pisot-Chabauty numbers. The theory of beta-expansions of p -adic numbers involves techniques from the theory of beta-expansions of real numbers as well as formal Laurent series.

The article is organised as follows: in Section 2, we introduce some basic definitions and preliminary results. In Section 3, we define the beta-expansions for p -adic numbers. In Section 4, we describe the periodic beta-expansions. The main result of this section is Theorem 4.1. In Section 5, we characterise the set finite expansions in Theorem 5.4.

2. BASIC DEFINITIONS AND RESULTS

Let p be a prime and

$$\mathbb{A}_p := \{mp^n : m, n \in \mathbb{Z}\} = \mathbb{Z}[\frac{1}{p}].$$

Then $\mathbb{A}_p \subset \mathbb{Q}$ is a principal ring. The unit group of \mathbb{A}_p is $\{p^k : k \in \mathbb{Z}\}$ and the field of fractions is \mathbb{Q} . Define

$$\nu_p : \mathbb{A}_p \rightarrow \mathbb{Z} \cup \{\infty\}$$

by

$$\nu_p(x) = \begin{cases} \max\{n \in \mathbb{Z} : p^n | x\}, & \text{if } x \neq 0; \\ \infty, & \text{if } x = 0. \end{cases}$$

Then ν_p verifies the properties

$$(2.1) \quad \nu_p(0) = \infty, \quad \nu_p(xy) = \nu_p(x) + \nu_p(y)$$

and

$$(2.2) \quad \begin{aligned} \nu_p(x + y) &\geq \min\{\nu_p(x), \nu_p(y)\} \quad \text{with} \\ \nu_p(x + y) &= \min\{\nu_p(x), \nu_p(y)\}, \quad \text{if } \nu_p(x) \neq \nu_p(y). \end{aligned}$$

Therefore $\nu_p(\cdot)$ is an exponential valuation on \mathbb{A}_p (*cf.* [16, Chapter II]). The p -adic norm $|\cdot|_p$ on \mathbb{A}_p is defined by

$$(2.3) \quad |x|_p := \begin{cases} p^{-\nu_p(x)}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

From (2.1) follows

$$(2.4) \quad |xy|_p = |x|_p |y|_p$$

and

$$(2.5) \quad \begin{aligned} |x+y|_p &\leq \max\{|x|_p, |y|_p\} \quad \text{with} \\ |x+y|_p &= \max\{|x|_p, |y|_p\}, \quad \text{if } |x|_p \neq |y|_p. \end{aligned}$$

Thus, $|\cdot|_p$ is a non-archimedean absolute value on \mathbb{A}_p . Let $|\cdot|_\infty$ be the archimedean absolute value. Then $|x|_p$ and $|x|_\infty$ satisfy the following product formula

$$\prod_{p \in \mathbb{P} \cup \{\infty\}} |x|_p = 1 \quad \text{for all } x \in \mathbb{Q} \setminus \{0\}$$

where \mathbb{P} denotes the set of primes. The completion of \mathbb{A}_p with respect to $|\cdot|_p$ is the field \mathbb{Q}_p of p -adic numbers. Thus

$$\mathbb{Z} \subset \mathbb{A}_p \subset \mathbb{Q} \subset \mathbb{Q}_p.$$

Each element $x \in \mathbb{Q}_p$ admits a unique expansion of the form

$$(2.6) \quad x = \sum_{n=n_0}^{\infty} x_n p^n, \quad \text{such that } n_0 \in \mathbb{Z}, \quad x_{n_0} \neq 0 \quad \text{and} \quad x_n \in \{0, \dots, p-1\}.$$

For expansions of the form (2.6), we will use the notation

$$x = \cdots p_2 p_1 p_0 \bullet p_{-1} \cdots p_{n_0}.$$

The x_i are called the p -adic coefficients of x . If we define the extension $\nu_p : \mathbb{Q}_p \rightarrow \mathbb{Z} \cup \{\infty\}$ by

$$\nu_p(x) := \begin{cases} n_0, & \text{if } x \neq 0; \\ \infty, & \text{if } x = 0, \end{cases}$$

then (2.3) holds also in \mathbb{Q}_p . The ring

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$$

is called the ring of p -adic integers. It easily follows that

$$\mathbb{Z} = \mathbb{A}_p \cap \mathbb{Z}_p = \{x \in \mathbb{A}_p : |x|_p \leq 1\}.$$

Furthermore \mathbb{Z}_p is compact. The elements of \mathbb{Z}_p can be expressed uniquely in the form (2.6) with $n_0 \geq 0$.

Definition 2.1. Each $x \in \mathbb{Q}_p$ of the form (2.6) has a unique *Artin decomposition*

$$x = [x]_p + \{x\}_p$$

with

$$[x]_p := \sum_{n \geq 0} x_n p^n \quad \text{and} \quad \{x\}_p := \sum_{n < 0} x_n p^n.$$

The number $[x]_p \in \mathbb{Z}_p$ is called *p -adic integer part* and $\{x\}_p \in \mathbb{A}_p \cap [0, 1)$ is called *p -adic fractional part* of x .

Remark 2.2. For $x \in \mathbb{R}$, let $\lfloor x \rfloor := \max\{k \in \mathbb{Z} : k \leq x\}$ be the real floor function. If $x \in \mathbb{A}_p$, it follows that $\lfloor x \rfloor_p = \lfloor x \rfloor$. However, for $x \in \mathbb{Q} \setminus \mathbb{A}_p$, this identity generally is not true. In any case, for $x \in \mathbb{R} \setminus \mathbb{Q}$ or $x \in \mathbb{Q}_p \setminus \mathbb{Q}$, one of these functions is not defined and therefore, this identity does not hold.

Definition 2.3. An element α is called *algebraic over \mathbb{A}_p* , if there is a polynomial

$$(2.7) \quad f(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{A}_p[x] \quad \text{with} \quad f(\alpha) = 0.$$

If f is irreducible over \mathbb{A}_p , then f is called a minimal polynomial of α . If $a_n = p^k$ for some $k \in \mathbb{Z}$, then α is called an *algebraic integer*. Since p^k is a unit of \mathbb{A}_p , we can assume without loss of generality, that $a_n = 1$.

It turns out that algebraic elements over \mathbb{A}_p are not necessarily contained in \mathbb{Q}_p . In our context, we will only need that $|\cdot|_p$ and $v_p(\cdot)$ can be extended uniquely from \mathbb{Q}_p to all of its algebraic extensions. This follows from the next

Theorem 2.4. [16, Chapter II, Theorem 4.8]. *Let K be a field which is complete with respect to $|\cdot|$ and L/K be an algebraic extension of degree m . Then $|\cdot|$ has a unique extension to L defined by*

$$|\alpha| = \sqrt[m]{|N_{L/K}(\alpha)|},$$

and L is complete with respect to this extension.

We apply Theorem 2.4 to algebraic extensions of \mathbb{Q}_p . Since \mathbb{Q}_p is complete, $|\cdot|_p$ and $v_p(\cdot)$ can be extended uniquely to each algebraic extension field L of $K = \mathbb{Q}_p$. Thus, every algebraic element over \mathbb{A}_p can be valued.

Let α be algebraic over \mathbb{A}_p and (2.7) be its minimal polynomial. The *Newton polygon of f* (cf. [16, Chapter II]) is defined as the lower convex hull of the set of points

$$\{(0, v_p(a_0)), \dots, (m, v_p(a_m))\}.$$

The polygon is a sequence of line segments E_1, E_2, \dots, E_r with monotonically increasing slopes.

Proposition 2.5. *Let*

$$f(x) = a_0 + \cdots + a_nx^n, \quad a_0a_n \neq 0,$$

be a polynomial over the field K , and K be complete with respect to the exponential valuation v . Let w be the unique extension of v to the splitting field L of f .

(i) *If*

$$\overline{(r, v(a_r))(s, v(a_s))}$$

is a line segment of slope $-m$ occurring in the Newton polygon of f , then $f(x)$ has exactly $s - r$ roots $\alpha_1, \dots, \alpha_{s-r}$ of value

$$w(\alpha_1) = \cdots = w(\alpha_{s-r}) = m.$$

- (ii) Let E_1, \dots, E_t be the line segments of the Newton polygon. Then, for each E_j , there exists a unique polynomial $g_j(x) \in K[x]$, such that

$$f(x) = a_n \prod_{j=1}^t g_j(x).$$

Thus,

$$g_j(x) = \prod_{w(\alpha_i)=m_j} (x - \alpha_i).$$

Remark 2.6. In this article, we will use the convention that, for algebraic elements α over \mathbb{A}_p , we will denote by $\alpha_1, \dots, \alpha_n$ the non-archimedean conjugates and by $\alpha_{n+1}, \dots, \alpha_{2n}$ the archimedean conjugates of α .

Lemma 2.7. Let $A \subset \mathbb{A}_p$. If A is bounded with respect to $|\cdot|_p$ and $|\cdot|_\infty$, i.e.

$$\max_{a \in A} |a|_p < \infty \quad \text{and} \quad \max_{a \in A} |a|_\infty < \infty,$$

then A is finite.

Proof. If $|a|_p \leq K$ for all $a \in A$, then

$$A \subset \left\{ mp^k : m \in \mathbb{Z}, k = \left\lfloor -\frac{\log K}{\log p} \right\rfloor \right\}.$$

Therefore, if A is bounded with respect to $|\cdot|_\infty$, it can contain only finitely many points. \square

Definition 2.8. A Pisot-Chabauty number (for short PC number) is a p -adic number $\alpha \in \mathbb{Q}_p$, such that

- (i) $\alpha_1 := \alpha$ is an algebraic integer over \mathbb{A}_p .
- (ii) $|\alpha_1|_p > 1$ for one non-archimedean conjugate of α .
- (iii) $|\alpha_i|_p \leq 1$ for all non-archimedean conjugates α_i , $i \in \{2, \dots, n\}$ of α .
- (iv) $|\alpha_i|_\infty < 1$ for all archimedean conjugates α_i , $i \in \{n+1, \dots, 2n\}$ of α .

If condition (iv) is replaced by

- (iv)' $|\alpha_i|_\infty = 1$ for all archimedean conjugates α_i , $i \in \{n+1, \dots, 2n\}$ of α ,

then α is called a Salem-Chabauty number (for short SC number).

Remark 2.9. If an archimedean root is located on the complex unit circle, then its minimal polynomial f is self-reciprocal, i.e. $f(x) = x^n f(\frac{1}{x})$. If there does not exist any root outside the unit circle, there also can not exist any root inside the unit circle. Therefore, condition (iv)' is formulated with equality for all archimedean conjugates of α .

Proposition 2.10. Let α be a PC number or SC number. Then $\alpha \in \mathbb{Q}_p$.

Proof. Let α and the minimal polynomial f of α be of the form (2.7). Since there is no non-archimedean conjugate α_j , $j \in \{2, \dots, n\}$ with $|\alpha|_p = |\alpha_j|_p$, the Newton polygon of f must contain an edge

$$\overline{(i, v_p(a_i))(i+1, v_p(a_{i+1}))},$$

with slope $\nu_p(\alpha) = v_p(a_{i+1}) - v_p(a_i)$ for some $i \in \{0, \dots, n-1\}$. By Proposition 2.5 (i), the minimal polynomial must contain the factor $x - \alpha \in \mathbb{Q}_p[x]$. Thus $\alpha \in \mathbb{Q}_p$. \square

Definition 2.11. Let

$$\mathcal{E}_n := \{(r_1, \dots, r_n) \in \mathbb{R}^n : x^n + r_n x^{n-1} + \dots + r_1 \text{ has only complex roots } \alpha \text{ with } |\alpha| < 1\}.$$

Then \mathcal{E}_n is an open set with $(0, \dots, 0) \in \mathcal{E}_n$. By considering the Newton polygon of the minimal polynomial, the following necessary conditions can be derived.

Proposition 2.12. *Let α be an algebraic integer over \mathbb{A}_p and (2.7) be its minimal polynomial with $a_n = 1$. Then α is a PC number if and only if*

$$(2.8) \quad \nu_p(a_{n-1}) \leq \min_{0 \leq j \leq n-2} \nu_p(a_j) \quad \text{and} \quad (a_0, \dots, a_{n-1}) \in \mathcal{E}_n.$$

Moreover, α is a SC number if and only if

$$(2.9) \quad \nu_p(a_{n-1}) \leq \min_{0 \leq j \leq n-2} \nu_p(a_j) \quad \text{and} \quad (a_0, \dots, a_{n-1}) \in \partial \mathcal{E}_n.$$

In both cases, α is an isolated root with $\alpha \in \mathbb{Q}_p$ and $|\alpha|_p = |a_{n-1}|_p$.

Proof. The Proposition follows directly from Proposition 2.5 (ii) and Definition 2.8. \square

Example 2.13. The PC numbers of degree one admit the form $\{x \in \mathbb{A}_p : |x|_p > 1\}$. In order to obtain PC numbers of arbitrary degree, the following construction may be used. We consider an irreducible polynomial

$$f(x) := p^k x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \in \mathbb{Z}[x]$$

such that

$$\nu_p(a_{n-1}) \leq \min_{0 \leq j \leq n-2} \nu_p(a_j)$$

and k is large enough such that

$$\left(\frac{a_{n-1}}{p^k}, \dots, \frac{a_0}{p^k} \right) \in \mathcal{E}_n.$$

Such a k exists, since $(0, \dots, 0)$ is an inner point of \mathcal{E}_n . Then the archimedean roots of f fulfill $|\alpha_i|_\infty < 1$. Since p^k is a unit of \mathbb{A}_p , the roots of f are algebraic integers over \mathbb{A}_p .

By considering the Newton polygon of f , we confirm that f has one non-archimedean root α_1 with $|\alpha_1|_p > 1$ and $n-1$ non-archimedean roots α_i with $|\alpha_i|_p \leq 1$. Therefore, α_1 is a PC number.

Remark 2.14. In the context of the present article, it turns out to be more convenient to write the minimal polynomial of PC numbers or SC numbers in the form

$$(2.10) \quad x^n - a_1 x^{n-1} - \dots - a_n, \quad a_i \in \mathbb{A}_p[x].$$

From now on, we will use this notation for the rest of the article.

In Theorem 2.16, a method to compute the p -adic coefficients of PC or SC numbers is given. For the proof, we will need the following auxiliary result.

Lemma 2.15. *If $z, w \in \mathbb{Q}_p$ with $|z|_p = |w|_p$, then $|z^n - w^n|_p \leq |z - w|_p |z|_p^{n-1}$ for all $n \in \mathbb{Z}$.*

Proof. The statement is trivial for $n = 0$. For $n > 0$, we have

$$\begin{aligned} |z^n - w^n|_p &= |z - w|_p |z^{n-1} + z^{n-2}w + \cdots + w^{n-1}|_p \\ &\leq |z - w|_p \max_{0 \leq j \leq n-1} |z^{n-1-j}w^j|_p \\ &= |z - w|_p |z|_p^{n-1} \end{aligned}$$

and

$$\begin{aligned} |z^{-n} - w^{-n}|_p &= |w^n - z^n|_p |z^{-n}w^{-n}|_p \\ &\leq |w - z|_p |w|_p^{n-1} |z^{-n}w^{-n}|_p \\ &= |z - w|_p |z|_p^{-n-1}. \end{aligned} \quad \square$$

Theorem 2.16. *Let α be a PC number or SC number and (2.10) be its minimal polynomial. Then the recurrence*

$$(2.11) \quad \begin{aligned} \alpha_0 &:= a_1, \\ \alpha_{k+1} &:= a_1 + \frac{a_2}{\alpha_k} + \cdots + \frac{a_n}{\alpha_k^{n-1}} \quad \text{for } k \geq 1 \end{aligned}$$

converges to

$$\lim_{k \rightarrow \infty} \alpha_k = \alpha.$$

Proof. First we prove by induction that $|\alpha_k|_p = |a_1|_p$ for all $k \geq 0$. For $k = 0$ this assertion follows from the definition. Let $|\alpha_k|_p = |a_1|_p$. For $j = 2, \dots, n$, it follows from Proposition 2.12 that $|a_1|_p > 1$ and $|a_1|_p \geq |a_j|_p$. Using (2.5), we easily obtain

$$\left| \frac{a_j}{\alpha_k^{j-1}} \right|_p = \frac{|a_j|_p}{|\alpha_k|_p^{j-1}} \leq \frac{|a_1|_p}{|a_1|_p} < |a_1|_p.$$

Thus $|\alpha_{k+1}|_p = |a_1|_p$. From Lemma 2.15, we get

$$\begin{aligned} |\alpha_{k+1} - \alpha_k|_p &= \left| a_2 \left(\frac{1}{\alpha_k} - \frac{1}{\alpha_{k-1}} \right) + \cdots + a_n \left(\frac{1}{\alpha_k^{n-1}} - \frac{1}{\alpha_{k-1}^{n-1}} \right) \right|_p \\ &\leq \max \left(\frac{|a_2|_p}{|a_1|_p^2}, \dots, \frac{|a_n|_p}{|a_1|_p^n} \right) |\alpha_k - \alpha_{k-1}|_p. \end{aligned}$$

Since $|a_1|_p > 1$ and $|a_1|_p \geq |a_j|_p$, the left factor is constant and less than 1. By Banach's fixed-point theorem, the sequence converges to a limit α with

$$\alpha = a_1 + \frac{a_2}{\alpha} + \cdots + \frac{a_n}{\alpha^{n-1}}. \quad \square$$

Remark 2.17. Proposition 2.10 follows also as a direct consequence from Theorem 2.16. This gives an alternative proof of Proposition 2.10.

Example 2.18. Let $f(x) = x^2 + \frac{1}{2}x + \frac{1}{2}$. Then f has two non-archimedean roots β_1, β_2 with $|\beta_1|_2 = 2$, $|\beta_2|_2 = 1$ and two archimedean roots β_3, β_4 with $|\beta_3|_\infty = |\beta_4|_\infty = \frac{1}{\sqrt{2}}$. Thus, the dominant non-archimedean root $\beta := \beta_1$ is a PC number. By Theorem 2.16, the recurrence (2.11) converges to

$$\beta = \cdots 110100010010011100011000110110011100111111010010 \bullet 1.$$

3. BETA-EXPANSIONS OF p -ADIC NUMBERS

Let $\beta \in \mathbb{Q}_p$ with $|\beta|_p > 1$ and $\alpha \in \mathbb{Z}_p$. A *representation in base β (or beta-representation)* of α is an infinite sequence $(d_i)_{i \geq 1}$, $d_i \in \mathbb{A}_p$ such that

$$(3.1) \quad \alpha = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i}.$$

A particular beta-representation – called the *beta-expansion* – can be computed by the following. This algorithm works as follows. Set $r^{(0)} = \alpha$ and let

$$d_k = \{\beta r^{(k-1)}\}_p, \quad r^{(k)} = \lfloor \beta r^{(k-1)} \rfloor_p$$

for $j \geq 1$. If

$$\mathcal{N} = [0, 1) \cap \{x \in \mathbb{A}_p : |x|_p \leq |\beta|_p\},$$

then $d_k \in \mathcal{N}$ for all $k \geq 1$. The set \mathcal{N} is called the digit set of the beta-expansion. It is finite with $|\beta|_p$ elements. Furthermore, $r^{(k)} \in \mathbb{Z}_p$ for all $k \geq 1$. The above procedure defines a mapping

$$d_\beta : \mathbb{Z}_p \rightarrow \mathcal{N}^{\mathbb{N}}$$

from \mathbb{Z}_p to $\mathcal{N}^{\mathbb{N}}$, the set of one-sided infinite sequences over \mathcal{N} , by

$$d_\beta(\alpha) := \bullet d_1 d_2 \cdots .$$

It follows that

$$r^{(k)} = \beta^k \left(\alpha - \sum_{i=1}^k d_i \beta^{-i} \right).$$

An equivalent definition is obtained by using the beta-transformation $T : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$, which is given by the mapping $z \mapsto \lfloor \beta z \rfloor_p$. For $k \geq 0$, define

$$T^0(x) := x \quad \text{and} \quad T^k(x) := T(T^{k-1}(x)).$$

Then $d_\beta(\alpha) = (d_k)_{k=1}^\infty$ if and only if $d_k = \{\beta T^{k-1}(\alpha)\}_p$ for all $k \geq 1$. If s is the one sided shift defined on $\mathcal{N}^{\mathbb{N}}$, i.e.

$$s(\bullet d_1 d_2 \cdots) := \bullet d_2 d_3 \cdots ,$$

we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathbb{Z}_p & \xrightarrow{T} & \mathbb{Z}_p \\ d_\beta \downarrow & & \downarrow d_\beta \\ \mathcal{N}^{\mathbb{N}} & \xrightarrow{s} & \mathcal{N}^{\mathbb{N}}. \end{array}$$

By iteration, we obtain $d_\beta(T^k(\alpha)) = s^k(d_\beta(\alpha))$ for all $k \geq 0$. Now let $\alpha \in \mathbb{Q}_p$ with $|\alpha|_p > 1$. Then there is an integer $n > 0$ such that

$$|\beta|_p^{n-1} < |\alpha|_p \leq |\beta|_p^n.$$

If

$$d_\beta(\beta^{-n}\alpha) = \bullet d_{-n} \cdots d_{-1} d_0 d_1 \cdots,$$

we define

$$d_\beta(\alpha) = d_{-n} \cdots d_{-1} d_0 \bullet d_1 d_2 \cdots.$$

Definition 3.1. Let

$$\mathbf{Per}(\beta) := \{\alpha \in \mathbb{Z}_p : d_\beta(\alpha) \text{ is eventually periodic}\} \quad \text{and}$$

$$\mathbf{Fin}(\beta) := \{\alpha \in \mathbb{Z}_p : d_\beta(\alpha) \text{ is finite}\}.$$

Then, clearly

$$\mathbf{Fin}(\beta) \subset \mathbf{Per}(\beta).$$

4. PERIODIC BETA-EXPANSIONS

The following theorem provides a p -adic analogue of the famous Theorem of Bertrand and Schmidt (*cf.* [8, 22]).

Theorem 4.1. *Let β be a PC number. Then $\mathbf{Per}(\beta) = \mathbb{Q}(\beta) \cap \mathbb{Z}_p$.*

Proof. The proof for $\mathbf{Per}(\beta) \subset \mathbb{Q}(\beta) \cap \mathbb{Z}_p$ is trivial. Therefore, we only prove the opposite inclusion $\mathbb{Q}(\beta) \cap \mathbb{Z}_p \subset \mathbf{Per}(\beta)$.

Let $z \in \mathbb{Q}(\beta) \cap \mathbb{Z}_p$. First we prove that the orbit of z under T is bounded with respect to $|\cdot|_p$ and $|\cdot|_\infty$. Let $r_1^{(0)} := z$, $r_1^{(k)} := T^k(z)$ and $r_j^{(k)}$ be the corresponding conjugates (with the numeration defined as in Remark 2.6). If $(d_k)_{k \geq 1}$ is the beta-expansion of $r_1^{(0)}$, then

$$(4.1) \quad r_j^{(k)} = \beta_j^k \left(r_j^{(0)} - \sum_{i=1}^k d_i \beta_j^{-i} \right) \quad \text{for all } j \in \{1, \dots, 2n\}.$$

Since $r_1^{(k)} \in \mathbb{Z}_p$ for all $k \geq 0$, it follows trivially that $|r_1^{(k)}|_p \leq 1$.

We consider the non-archimedean conjugates $r_j^{(k)}$, $j \in \{2, \dots, n\}$. Since

$$|\beta_j|_p \leq 1 \quad \text{and} \quad |d_\ell|_p \leq |\beta|_p,$$

it follows by (2.2),

$$(4.2) \quad \begin{aligned} |r_j^{(k)}|_p &\leq \max \left(|\beta_j^k r_j^{(0)}|_p, \max_{1 \leq i \leq k} (|d_i \beta_j^{k-i}|_p) \right) \\ &\leq \max \left(|r_j^{(0)}|_p, |\beta|_p \right) < \infty. \end{aligned}$$

Therefore, $|r_j^{(k)}|_p$ is bounded for all k and j .

Now we consider the archimedean conjugates $r_j^{(k)}$, $j \in \{n+1, \dots, 2n\}$. Let

$$\gamma := \max_{n+1 \leq j \leq 2n} |\beta_j|_\infty.$$

Since

$$\gamma < 1 \quad \text{and} \quad |d_i|_\infty < 1,$$

it follows from (4.1) that

$$|r_j^{(k)}|_\infty \leq \gamma^k |r_j^{(0)}|_\infty + \sum_{i=1}^k \gamma^{k-i} < \infty.$$

We need a technical result.

Lemma 4.2. *Define the matrices*

$$B_p := (\beta_j^{-i}) \in \mathbb{Q}_p^{n \times n} \quad \text{and} \quad B_\infty := (\beta_{n+j}^{-i}) \in \mathbb{C}^{n \times n}$$

with $i \in \{0, \dots, n-1\}$ and $j \in \{1, \dots, n\}$. If

$$R_p^{(k)} := (r_1^{(k)}, \dots, r_n^{(k)}), \quad R_\infty^{(k)} := (r_{n+1}^{(k)}, \dots, r_{2n}^{(k)}),$$

then for every $k \geq 0$, there exists a unique n -tuple

$$W^{(k)} := (w_0^{(k)}, \dots, w_{n-1}^{(k)}) \in \mathbb{A}_p^n$$

such that

$$R_p^{(k)} = q^{-1} W^{(k)} B_p \quad \text{and} \quad R_\infty^{(k)} = q^{-1} W^{(k)} B_\infty$$

with $q \in \mathbb{N}$.

Proof. The Lemma is proved by induction. For $k = 0$, it follows from

$$\beta_j^n = a_1 \beta_j^{n-1} + \dots + a_n,$$

that

$$(4.3) \quad r_j^{(0)} = q^{-1} \sum_{i=0}^{n-1} z_i \beta_j^i = q^{-1} \sum_{i=1}^n w_i^{(0)} \beta_j^{-i}.$$

If $k > 0$, then

$$(4.4) \quad \begin{aligned} r_j^{(k)} &= \beta_j r_j^{(k-1)} - d_k \\ &= q^{-1} \left(\beta_j \sum_{i=1}^n w_i^{(k-1)} \beta_j^{-i} - q d_k \right) \\ &= q^{-1} \left(w_1^{(k-1)} - q d_k + \sum_{i=1}^{n-1} w_{i+1}^{(k-1)} \beta_j^{-i} \right) \\ &= q^{-1} \left(\sum_{i=1}^n w_i^{(k)} \beta_j^{-i} \right). \end{aligned}$$

The Lemma follows from the definition of the matrices B_p and B_∞ . □

We continue now with the proof of Theorem 4.1. On \mathbb{A}_p^n , we define the vector norms

$$\|(a_1, \dots, a_n)\|_p := \max_{1 \leq i \leq n} |a_i|_p \quad \text{and} \quad \|(a_1, \dots, a_n)\|_\infty := \max_{1 \leq i \leq n} |a_i|_\infty.$$

The induced matrix norms on $\mathbb{A}_p^{n \times n}$ are given by

$$\|A\|_p = \max_{1 \leq i, j \leq n} |a_{ij}|_p \quad \text{and} \quad \|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|_\infty.$$

Since B_p and B_∞ are invertible, and the $r_j^{(k)}$ are bounded for all $j \in \{1, \dots, 2n\}$, it follows by Lemma 4.2 that

$$\begin{aligned} \max_{0 \leq j \leq n-1} |w_j^{(k)}|_p &= \|W^{(k)}\|_p = \|qR_p^{(k)}B_p^{-1}\|_p \\ &\leq \|R_p^{(k)}\|_p \|qB_p^{-1}\|_p < \infty \end{aligned}$$

and analogously,

$$\max_{0 \leq j \leq n-1} |w_j|_\infty < \infty.$$

Since $w_j \in \mathbb{A}_p$ for all j , by Lemma 2.7, the w_j must be contained in a finite subset of \mathbb{A}_p . Therefore, the $r_1^{(k)}$ must be contained in a finite subset of $\mathbb{Q}(\beta)$ and thus, there exists an m such that $r_1^{(k+m)} = r_1^{(k)}$ for all k large enough. This implies that the beta-expansion of z is eventually periodic. \square

Example 4.3. Since $\beta = \frac{1}{p}$ is the root of $f(x) = px - 1$, one easily verifies that $\frac{1}{p}$ is a PC number. Since $\mathbb{Q}(\frac{1}{p}) = \mathbb{Q}$, Theorem 4.1 implies that the numbers of $\mathbb{Q} \cap \mathbb{Z}_p$ admit eventually periodic p -adic expansions.

Theorem 4.4. *Let $S \subset \mathbb{N}$ be an infinite set of positive integers, such that $1 \in S$. If $S \subset \mathbf{Per}(\beta)$, then β is a PC number or SC number.*

Proof. Since $1 \in S \subset \mathbf{Per}(\beta)$, it follows that $d_\beta(1) = \bullet d_1 d_2 \dots$ is eventually periodic. If k is the length of the preperiod and ℓ is the length of the period, then $d_{i+\ell} = d_i$ for all $i \geq k+1$. Therefore,

$$d_\beta(1) = \bullet d_1 \dots d_k d_{k+1} \dots d_{k+\ell} d_{k+1} \dots d_{k+\ell} \dots$$

and thus,

$$(4.5) \quad (\beta^\ell - 1) \beta^k \left(1 - \frac{d_1}{\beta} - \dots - \frac{d_k}{\beta^k} \right) - \beta^\ell \left(\frac{d_{k+1}}{\beta} + \dots + \frac{d_{k+\ell}}{\beta^\ell} \right) = 0.$$

Note that in the case that $d_\beta(1)$ is finite, the second summand of (4.5) is zero. Since $d_i \in \mathbb{A}_p$, it follows that β is an algebraic integer over \mathbb{A}_p .

We consider the Newton polygon of (4.5). Since $d_1 = \{\beta \cdot 1\}_p$, it follows that

$$\nu_p(1) = 0, \quad \nu_p(-d_1) = \nu_p(\beta) < 0 \quad \text{and} \quad \nu_p(d_j) \geq \nu_p(\beta) \quad \text{for} \quad j \geq 2.$$

Therefore, the Newton polygon contains one edge with slope $-\nu_p(\beta) > 0$ and all other edges have slopes ≤ 0 . By Proposition 2.5 (i), there exists one non-archimedean root α of (4.5) with $\nu_p(\alpha) = \nu_p(\beta) < 0$ and all other non-archimedean roots $\tilde{\alpha}$ have $\nu_p(\tilde{\alpha}) \geq 0$. Since (4.5) is a multiple of the minimal polynomial of β , it follows that $\alpha = \beta$ and $|\tilde{\alpha}| \leq 1$.

Now we examine the archimedean conjugates of β . Assume that there exists an archimedean conjugate β_j , $j \in \{n+1, \dots, 2n\}$, such that $|\beta_j|_\infty > 1$. Since S is infinite, there exists a $k \in S$, with

$$(4.6) \quad k > \frac{1}{|\beta_j|_\infty - 1}.$$

It follows that

$$k = \sum_{i=1}^s \frac{e_i}{\beta^i} + \frac{r_1^{(s)}}{\beta^s} = \sum_{i=1}^s \frac{e_i}{\beta_j^i} + \frac{r_j^{(s)}}{\beta_j^s} \quad \text{for all } s \geq 0.$$

Since $k \in \mathbf{Per}(\beta)$, the sequence $d_\beta(k) = \bullet e_1 e_2 \dots$ is eventually periodic and thus, the $r_1^{(s)}$ and $r_j^{(s)}$ can take only finitely many values. Therefore, the sequence

$$k = \sum_{i=1}^{\infty} \frac{e_i}{\beta_j^i}$$

converges. Since $|e_i|_\infty < 1$, it follows that

$$k \leq \frac{1}{|\beta_j|_\infty - 1},$$

which contradicts (4.6). □

5. SHIFT RADIX SYSTEMS AND FINITE BETA-EXPANSIONS

Shift radix systems, were introduced in [2] in order to provide a unified notation for two well known types of number systems, namely, canonical number systems as well as beta-expansions of real numbers.

Let $n \geq 1$ be an integer and $\mathbf{r} = (r_1, \dots, r_n) \in \mathbb{R}^n$. Let $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the real floor and ceiling functions respectively. To the vector \mathbf{r} , we associate the mapping $\tilde{\tau}_{\mathbf{r}} : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ as follows: if $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{Z}^n$ then let

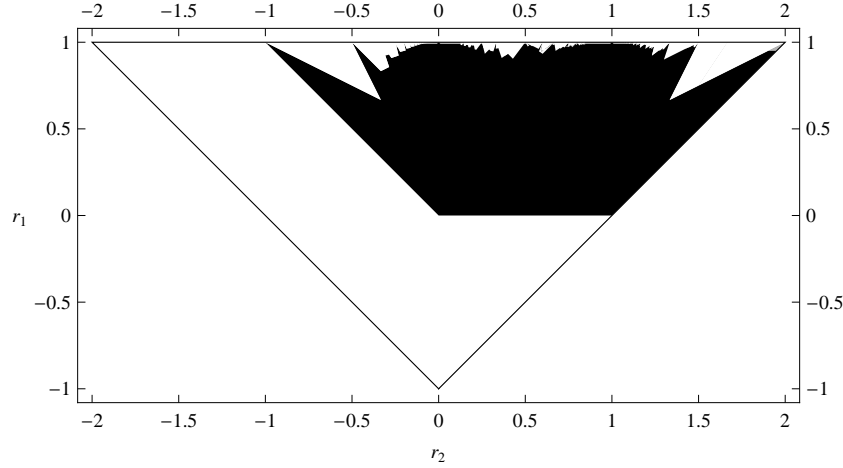
$$(5.1) \quad \tilde{\tau}_{\mathbf{r}}(\mathbf{z}) := (z_2, \dots, z_n, -\lfloor \mathbf{r}\mathbf{z} \rfloor),$$

where $\mathbf{r}\mathbf{z}$ is the euclidian inner product of \mathbf{r} and \mathbf{z} , i.e.

$$\mathbf{r}\mathbf{z} = r_1 z_1 + \dots + r_n z_n.$$

Then $(\mathbb{Z}^n, \tilde{\tau}_{\mathbf{r}})$ is called a *shift radix system* (for short SRS) on \mathbb{Z}^n . Note that $\mathbf{0} = (0, \dots, 0)$ is a fixed point of $\tilde{\tau}_{\mathbf{r}}$. We say that the orbit of \mathbf{z} under $\tilde{\tau}_{\mathbf{r}}$ ends up in $\mathbf{0}$, if there exists a $k \geq 0$ such that $\tilde{\tau}_{\mathbf{r}}^k(\mathbf{z}) = \mathbf{0}$.

It is an important problem for canonical number systems as well as beta-expansions of real numbers to determine whether each number admits a finite expansion. In SRS language, this translates to the following question:

FIGURE 1. The set \mathcal{D}_2^0 (in black) inside \mathcal{D}_2 (the interior of the triangle).

For which $\mathbf{r} \in \mathbb{Z}^n$, we have that all orbits of $(\mathbb{Z}^n, \tilde{\tau}_{\mathbf{r}})$ end up in $\mathbf{0}$?

In the present section, we will prove that beta-expansions of p -adic numbers can be described by a slight variation of this dynamical system, namely by $(\mathbb{Z}^n, \tau_{\mathbf{r}})$ with

$$\tau_{\mathbf{r}}(\mathbf{z}) = (z_2, \dots, z_n, -\lceil \mathbf{r}\mathbf{z} \rceil).$$

In Proposition 5.1, we will prove that for a fixed given vector $\mathbf{r} \in \mathbb{R}^n$, both of these systems show exactly the same behaviour. Let

$$(5.2) \quad \begin{aligned} \mathcal{D}_n &:= \{\mathbf{r} \in \mathbb{R}^d : \forall \mathbf{z} \in \mathbb{Z}^d, \text{ the sequence } (\tilde{\tau}_{\mathbf{r}}^k(\mathbf{z}))_{k \geq 1} \text{ is eventually periodic}\}, \\ \mathcal{D}_n^0 &:= \{\mathbf{r} \in \mathbb{R}^d : \forall \mathbf{z} \in \mathbb{Z}^d, \exists k > 0 : \tilde{\tau}_{\mathbf{r}}^k(\mathbf{z}) = \mathbf{0}\}. \end{aligned}$$

In [2, 3], it is proved that the inclusions $\mathcal{D}_n^0 \subset \mathcal{D}_n$ and $\mathcal{E}_n \subset \mathcal{D}_n \subset \overline{\mathcal{E}_n}$ hold. It is easy to verify that $\mathcal{D}_1^0 = [0, 1]$ and $\mathcal{D}_1 = [-1, 1]$. However, a full description of \mathcal{D}_2^0 and \mathcal{D}_2 is already unknown. The sets \mathcal{D}_n^0 and \mathcal{D}_n have been extensively studied in the last years (cf. [2, 3, 4, 5, 7, 11, 14, 19, 24]). In Figure 1, an approximation of \mathcal{D}_2^0 is shown.

It is conjectured, that $\mathcal{D}_n^0 \subset \mathcal{E}_n$ which is equivalent to $\mathcal{D}_n^0 \cap \partial \mathcal{E}_n = \emptyset$ (cf. [3]). Up to now, this conjecture has been proved only for $n \leq 3$ (cf. [3, 11]). For a very recent survey on SRS, we refer to [15].

Proposition 5.1. *Let $\mathbf{r} \in \mathbb{R}^n$. All orbits of $(\mathbb{Z}^n, \tau_{\mathbf{r}})$ end up in $\mathbf{0}$ if and only if $\mathbf{r} \in \mathcal{D}_n^0$.*

Proof. At first we show that $\tau_{\mathbf{r}}(\mathbf{z}) = -\tilde{\tau}_{\mathbf{r}}(-\mathbf{z})$ for all $\mathbf{z} \in \mathbb{Z}^n$. Indeed, consider an arbitrary $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{Z}^n$ and note that for any real number x we have $\lceil x \rceil = -\lfloor -x \rfloor$. We obtain

$$\begin{aligned} \tau_{\mathbf{r}}(\mathbf{z}) &= (z_2, \dots, z_n, -\lceil \mathbf{r}\mathbf{z} \rceil) = (z_2, \dots, z_n, \lfloor -\mathbf{r}\mathbf{z} \rfloor) \\ &= -(-z_2, \dots, -z_n, -\lfloor -\mathbf{r}\mathbf{z} \rfloor) = -\tilde{\tau}_{\mathbf{r}}(-\mathbf{z}). \end{aligned}$$

By induction, it follows that

$$(5.3) \quad \tau_{\mathbf{r}}^k(\mathbf{z}) = -\tilde{\tau}_{\mathbf{r}}^k(-\mathbf{z}).$$

Now we easily see that all orbits of $(\mathbb{Z}^n, \tau_{\mathbf{r}})$ end up in $\mathbf{0}$ if and only if all orbits of $(\mathbb{Z}^n, \tilde{\tau}_{\mathbf{r}})$ do so. \square

In order to prove Theorem 5.4, which is the main result of this section, we need the following preliminary results.

Proposition 5.2. *If $U, V \in \mathbb{Q}_p$, then*

$$|U - \{U + V\}_p|_p = |[U + V]_p - V|_p = \begin{cases} |[U]_p|_p, & \text{if } |V|_p \leq 1; \\ |\{V\}_p|_p, & \text{if } |V|_p > 1; \end{cases}$$

or equivalently,

$$\nu_p(U - \{U + V\}_p) = \nu_p([U + V]_p - V) = \begin{cases} \nu_p([U]_p), & \text{if } \nu_p(V) \geq 0; \\ \nu_p(\{V\}_p), & \text{if } \nu_p(V) < 0. \end{cases}$$

Proof. Straight forward. \square

Proposition 5.3. *Let β be an arbitrary element of \mathbb{Q}_p with $|\beta|_p > 1$, and let $z \in \mathbb{A}_p[\beta^{-1}] \cap \mathbb{Z}_p$ have purely periodic beta-expansion with period ℓ . Then $z \in \mathbb{A}_p[\beta] \cap \mathbb{Z}_p$.*

Proof. Assume $z \in \mathbb{A}_p[\beta^{-1}] \cap \mathbb{Z}_p$ is purely periodic with period ℓ . Let $d_\beta(z) = \bullet d_1 d_2 \dots$. Since $z \in \mathbb{A}_p[\beta^{-1}]$, there is an m such that $\beta^{m\ell} z \in \mathbb{A}_p[\beta]$. Therefore

$$z = \beta^{m\ell} z - d_1 \beta^{m\ell-1} - \dots - d_{m\ell} \in \mathbb{A}_p[\beta]. \quad \square$$

Now we are able to state the main result of this section.

Theorem 5.4. *Let $\beta \in \mathbb{Q}_p$ with $|\beta|_p > 1$. Then*

$$(F) \quad \mathbf{Fin}(\beta) = \mathbb{A}_p[\beta^{-1}] \cap \mathbb{Z}_p,$$

if and only if β is an PC number and its minimal polynomial

$$(5.4) \quad x^n - a_1 x^{n-1} - \dots - a_n, \quad a_i \in \mathbb{A}_p[x]$$

fulfills

$$(5.5a) \quad \max_{2 \leq i \leq n} |a_i|_p < |a_1|_p \quad \text{and}$$

$$(5.5b) \quad -\mathbf{a} \in \mathcal{D}_n^0, \quad \text{where } \mathbf{a} := (a_n, \dots, a_1).$$

Proof. We will prove first that (5.5) implies (F). Since it is trivial that $\mathbf{Fin}(\beta) \subset \mathbb{A}_p[\beta^{-1}] \cap \mathbb{Z}_p$, we will prove only the opposite inclusion. The proof runs in two steps. Condition (5.5a) ensures that after some preliminary phase, the dynamical system reduces to a classical SRS. Condition (5.5b) ensures that all orbits of this SRS end up in $\mathbf{0}$.

Let $z \in \mathbb{A}_p[\beta^{-1}] \cap \mathbb{Z}_p$. From Theorem 4.1, it follows that

$$\mathbb{A}_p[\beta^{-1}] \cap \mathbb{Z}_p \subset \mathbb{Q}(\beta) \subset \mathbf{Per}(\beta).$$

Thus, z has an eventually periodic expansion. Without loss of generality, we can assume that $z \in \mathbb{A}_p[\beta]$. Otherwise, we replace z by an element from the orbit with purely periodic expansion and apply Proposition 5.3. Let

$$\mathcal{B} = \{1, \beta, \dots, \beta^{n-1}\} \quad \text{and} \quad \mathcal{V} = \{v_1, \dots, v_n\}$$

where

$$(5.6a) \quad v_j = \beta^{n-j} - a_1\beta^{n-j-1} - \dots - a_{n-j}$$

$$(5.6b) \quad = \frac{a_{n-j+1}}{\beta} + \dots + \frac{a_n}{\beta^j}$$

for $j \in \{1, \dots, n\}$. Note $v_n = 1$. Let $\mathbf{v} = (v_1, \dots, v_n)$. Then both \mathcal{B} and \mathcal{V} are two different bases of $\mathbb{A}_p[\beta]$ considered as a lattice over \mathbb{A}_p . Using (5.6), the coordinates with respect to \mathcal{V} can be computed from the coordinates with respect to \mathcal{B} by a linear system of equations. In this way, we define a bijection $\varphi : \mathbb{A}_p[\beta] \rightarrow \mathbb{A}_p^n$ by

$$\varphi(z) := (z_1, \dots, z_n).$$

By construction, φ^{-1} is given by

$$\varphi^{-1}((z_1, \dots, z_n)) = z_1 v_1 + \dots + z_n v_n.$$

In base \mathcal{V} , multiplication by β is represented by the matrix

$$M := \begin{pmatrix} 0 & \cdots & \cdots & 0 & a_n \\ 1 & \ddots & & \vdots & \vdots \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & a_2 \\ 0 & \cdots & 0 & 1 & a_1 \end{pmatrix}$$

such that

$$\varphi(\beta z) = \varphi(z)M.$$

Let $\mathbf{e} := \varphi(1) = (0, \dots, 0, 1)$. Since

$$T(z) = \beta z - \{\beta z\}_p,$$

the beta-transformation with respect to \mathcal{V} takes the form

$$(5.7) \quad \sigma : \mathbb{A}_p^n \rightarrow \mathbb{A}_p^n \quad \text{with} \quad \mathbf{z} \mapsto \mathbf{z}M - \{\mathbf{z}M\mathbf{v}^\top\}_p \mathbf{e}.$$

Then

$$\varphi(T(z)) = \sigma(\varphi(z)) \quad \text{and} \quad T(\varphi^{-1}(\mathbf{z})) = \varphi^{-1}(\sigma(\mathbf{z})),$$

which is indicated in the following commutative diagram:

$$\begin{array}{ccc} \mathbb{A}_p[\beta] & \xrightarrow{T} & \mathbb{A}_p[\beta] \\ \varphi^{-1} \uparrow \downarrow \varphi & & \varphi^{-1} \uparrow \downarrow \varphi \\ \mathbb{A}_p^n & \xrightarrow{\sigma} & \mathbb{A}_p^n \end{array}$$

Substituting M , \mathbf{v} and \mathbf{e} into (5.7), we can express the map σ as follows:

$$\begin{aligned} \sigma : (z_1, \dots, z_n) &\mapsto (z_2, \dots, z_{n+1}) \quad \text{with} \\ z_{n+1} &= a_n z_1 + \dots + a_1 z_n - \{a_n z_1 + \dots + a_1 z_n + v_1 z_2 + \dots + v_{n-1} z_n\}_p. \end{aligned}$$

The k th iterate of σ is given by

$$\sigma^k((z_1, \dots, z_n)) = (z_{k+1}, \dots, z_{k+n}),$$

where

$$z_{k+n} = U_k - \{U_k + V_k\}_p$$

with

$$(5.8) \quad \begin{aligned} U_k &:= a_n z_k + \dots + a_1 z_{k+n-1} \quad \text{and} \\ V_k &:= v_1 z_{k+1} + \dots + v_{n-1} z_{k+n-1}. \end{aligned}$$

From (5.5a), (5.6b) and $|a_1|_p = |\beta|_p$, it follows that

$$(5.9) \quad \max_{1 \leq i \leq n-1} |v_i|_p < 1$$

and therefore,

$$|V_k|_p \leq \max_{1 \leq i \leq n-1} |v_i z_{k+i}|_p < \max_{k+1 \leq i \leq k+n-1} |z_i|_p.$$

By Proposition 5.2, we get the following implications:

- If $|V_k|_p > 1$, it follows that

$$|z_{k+n}|_p = |V_k|_p < \max_{k+1 \leq i \leq k+n-1} |z_i|_p.$$

- If $|V_k|_p \leq 1$, it follows that

$$|z_{k+n}|_p = |[U_k]_p|_p \leq 1.$$

Therefore, there must exist some $k_0 \geq 1$ such that

$$(5.10) \quad \max_{0 \leq j \leq n-1} |z_{k+j}|_p \leq 1$$

for all $k > k_0$. Since $z_k \in \mathbb{A}_p$, it follows that

$$(5.11) \quad \sigma^k((z_1, \dots, z_n)) = (z_{k+1}, \dots, z_{k+n}) \in \mathbb{Z}^n$$

holds for all $k \geq k_0$. From (5.10), it follows by (5.9) that $|V_k|_p < 1$. Therefore,

$$\{U_k + V_k\}_p = \{U_k\}_p$$

and thus,

$$z_{k+n} = U_k - \{U_k + V_k\}_p = [U_k]_p.$$

Since $U_k \in \mathbb{A}_p$, it follows that

$$[U_k]_p = [U_k] = -[-U_k]$$

(cf. Remark 2.2). Thus

$$\begin{aligned} \sigma((z_k, \dots, z_{k+n-1})) &= (z_{k+1}, \dots, z_{k+n-1}, -[-U_k]) \\ &= \tau_{-\mathbf{a}}((z_k, \dots, z_{k+n-1})). \end{aligned}$$

By (5.3), it follows that

$$\begin{aligned}\sigma^k((z_1, \dots, z_n)) &= \tau_{-\mathbf{a}}^{k-k_0}(\sigma^{k_0}((z_1, \dots, z_n))) \\ &= -\tilde{\tau}_{-\mathbf{a}}^{k-k_0}(-\sigma^{k_0}((z_1, \dots, z_n)))\end{aligned}$$

for all $k \geq k_0$. Since $-\mathbf{a} \in \mathcal{D}_n^0$, there must exist some $k \geq k_0$ with

$$\sigma^k((z_1, \dots, z_n)) = \mathbf{0}.$$

Thus, each $z \in \mathbb{A}_p[\beta^{-1}] \cap \mathbb{Z}_p$ admits a finite representation with respect to β .

Now we will prove the converse direction of Theorem 5.4. Suppose that

$$\mathbb{A}_p[\beta^{-1}] \cap \mathbb{Z}_p = \mathbf{Fin}(\beta).$$

Since $\mathbb{N} \subset \mathbf{Fin}(\beta) \subset \mathbf{Per}(\beta)$, it follows by Theorem 4.4, that β is a PC number or SC number. Without loss of generality, we can assume that the minimal polynomial has the form (5.4). In order to exclude the case of SC numbers, we distinguish the following cases.

- (i) For $n = 1$, there do not exist any SC numbers.
- (ii) For $n = 2$, it is proved in [3], that $\mathcal{D}_2^0 \cap \partial\mathcal{E}_2 = \emptyset$. If there exists an archimedean root of (5.4) that is located on the complex unit circle, it follows that $-\mathbf{a} \in \partial\mathcal{E}_2$, which contradicts (5.5b).
- (iii) Let $n > 2$. If (5.4) is the minimal polynomial of a SC number, it must be self-reciprocal (cf. Remark 2.9). Then $a_1 = a_{n-1}$, which contradicts (5.5a).

Thus, β is a PC number. We will prove now that each of the converses of (5.5a) and (5.5b) contradict (F).

In order to prove the necessity of (5.5a), we follow the proofs of [21, Lemma 2.4] and [19, Theorem 5.4]. Suppose that

$$(5.12) \quad \max_{2 \leq i \leq n} |a_i|_p \geq |a_1|_p \quad \text{or equivalently} \quad \min_{2 \leq i \leq n} \nu_p(a_i) \leq \nu_p(a_1).$$

We will construct an element $z \in \mathbb{A}_p[\beta]$ that does not have a finite representation.

At first, we will prove that there exists some index $h \in \{1, \dots, n-1\}$, such that $\nu_p(v_h) \leq 0$. Define

$$(5.13) \quad h_0 := \max \left\{ h \in \{2, \dots, n\} : \nu_p(a_h) = \min_{2 \leq j \leq n} \nu_p(a_j) \right\},$$

i.e. h_0 is the minimal index $h \in \{2, \dots, n\}$ such that $\nu_p(a_h)$ attains its minimal value. Then, by (5.12) and (5.13), it follows that $2 \leq h_0 \leq n$ and thus

$$(5.14) \quad 1 \leq n - h_0 + 1 \leq n - 1.$$

By (5.13), it follows that $\nu_p(a_j) > \nu_p(a_{h_0})$ for all $j > h_0$ and thus, by (5.6),

$$\begin{aligned}\nu_p(v_{n-h_0+1}) &= \nu_p\left(\frac{a_{h_0}}{\beta} + \dots + \frac{a_n}{\beta^{n-h_0+1}}\right) \\ &= \nu_p(a_{h_0}) - \nu_p(\beta) \leq \nu_p(a_{h_0}) - \nu_p(a_1) \leq 0.\end{aligned}$$

If

$$(5.15) \quad i_0 := \min \left\{ i \in \{1, \dots, n-1\} : \nu_p(v_i) = \min_{1 \leq j \leq n-1} \nu_p(v_j) \right\},$$

then $\nu_p(v_{i_0}) \leq 0$. For $(z_1, \dots, z_n) \in \mathbb{A}_p^n$, define

$$\begin{aligned} j_0((z_1, \dots, z_n)) &:= \\ &:= \begin{cases} \max \left\{ i \in \{2, \dots, n\} : \nu_p(z_i) = \min_{i_0+1 \leq j \leq n} \nu_p(z_j) \right\}, & \text{if } \min_{i_0+1 \leq j \leq n} \nu_p(z_j) < 0; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence, if $j_0((z_1, \dots, z_n)) > 0$, it follows that $(z_1, \dots, z_n) \neq \mathbf{0}$. If

$$\mathbf{z}^{(0)} := (z_1^{(0)}, \dots, z_n^{(0)}) = (0, \dots, 0, p^{-1}),$$

then $j_0(\mathbf{z}^{(0)}) = n$ and thus,

$$(5.16) \quad i_0 + 1 \leq j_0(\mathbf{z}^{(0)}) \leq n.$$

For $k \geq 0$, let

$$\mathbf{z}^{(k)} := (z_1^{(k)}, \dots, z_n^{(k)}) = (z_{k+1}, \dots, z_{k+n}) = \sigma^k(\mathbf{z}^{(0)}).$$

We will show that $\mathbf{z}^{(0)}$ has an infinite representation by proving that

$$(5.17) \quad i_0 + 1 \leq j_0(\mathbf{z}^{(k)}) \leq n$$

for all $k \geq 0$ and thus,

$$\mathbf{z}^{(k)} = \sigma^k(\mathbf{z}^{(0)}) \neq \mathbf{0}.$$

We will prove (5.17) by induction. By (5.16), equation (5.17) holds for $k = 0$. Thus, we can proceed to the induction step. Suppose that (5.17) holds for $k-1 \geq 0$ and note that

$$(5.18) \quad \mathbf{z}^{(k)} = (z_1^{(k)}, \dots, z_n^{(k)}) = (z_2^{(k-1)}, \dots, z_n^{(k-1)}, z_n^{(k)}).$$

Let $j_0 := j_0(\mathbf{z}^{(k-1)})$. We distinguish the following cases.

(i) Let $j_0 > i_0 + 1$. By (5.18), it follows that

$$\begin{aligned} \min_{i_0+1 \leq j \leq n} \nu_p(z_j^{(k)}) &\leq \min\{\nu_p(z_{j_0}^{(k-1)}), \nu_p(z_n^{(k)})\} \\ &= \min\{\nu_p(z_{j_0-1}^{(k)}), \nu_p(z_n^{(k)})\} \quad (\text{by the definition of } j_0) \\ &< 0. \end{aligned}$$

Thus $j_0(\mathbf{z}^{(k)}) = j_0 - 1$ or $j_0(\mathbf{z}^{(k)}) = n$. Both of these inequalities imply that

$$(5.19) \quad i_0 + 1 \leq j_0(\mathbf{z}^{(k)}) \leq n.$$

(ii) Let $j_0 = i_0 + 1$. From (5.8) it follows that

$$\begin{aligned} U_k &:= a_n z_1^{(k-1)} + \dots + a_1 z_n^{(k-1)} \quad \text{and} \\ V_k &:= v_1 z_2^{(k-1)} + \dots + v_{n-1} z_n^{(k-1)}. \end{aligned}$$

The definitions of i_0 and j_0 imply that

$$\begin{aligned} \nu_p(v_{i_0}) &\leq 0, & \nu_p(v_i) &> \nu_p(v_{i_0}) && \text{for } i < i_0, \\ \nu_p(v_i) &\geq \nu_p(v_{i_0}) && \text{for } i \geq i_0, \end{aligned}$$

and

$$\begin{aligned} \nu_p(z_{j_0}^{(k-1)}) &< 0, & \nu_p(z_j^{(k-1)}) &> \nu_p(z_{j_0}^{(k-1)}) && \text{for } j > j_0, \\ \nu_p(z_j^{(k-1)}) &\geq \nu_p(z_{j_0}^{(k-1)}) && \text{for } j \leq j_0. \end{aligned}$$

Hence,

$$\nu_p(v_{i_0} z_{i_0+1}^{(k-1)}) = \nu_p(v_{i_0} z_{j_0}^{(k-1)}) < \nu_p(v_i z_{i+1}^{(k-1)}) \quad \text{for } i \neq i_0.$$

By (2.2), it follows that

$$\nu_p(V_k) = \nu_p(v_1 z_2^{(k-1)} + \cdots + v_{n-1} z_n^{(k-1)}) = \nu_p(v_{i_0} z_{i_0+1}^{(k-1)}) \leq \nu_p(z_{j_0}^{(k-1)}),$$

and therefore, by Proposition 5.2,

$$\nu_p(z_n^{(k)}) = \nu_p(U_k - \{U_k + V_k\}_p) = \nu_p(\{V_k\}_p) \leq \nu_p(z_{j_0}^{(k-1)}).$$

Thus,

$$(5.20) \quad i_0 + 1 \leq j_0(\mathbf{z}^{(k+1)}) \leq n$$

and we are done also in this case.

Therefore, the necessity of (5.5a) is proved.

Assume now that (F) holds but (5.5b) does not hold, i.e. $-\mathbf{a} \notin \mathcal{D}_n^0$. By (5.2), there exists a $\mathbf{z} \in \mathbb{Z}^n$ such that $\tilde{\tau}_{-\mathbf{a}}^k(\mathbf{z})$ does not end up in $\mathbf{0}$. Therefore,

$$\tau_{-\mathbf{a}}^k(-\mathbf{z}) = -\tilde{\tau}_{-\mathbf{a}}^k(\mathbf{z})$$

does not end up in $\mathbf{0}$. Thus, $\varphi^{-1}(-\mathbf{z}) \in \mathbb{A}_p[\beta]$ does not have a finite expansion. This is a contradiction to (F) and thus, (5.5b) is a necessary condition. \square

Remark 5.5. Condition (5.5b) indicates a relation to canonical number systems. In order to explain this connection in more detail, we recall Example 2.13. If the polynomial

$$\tilde{g}(x) = \tilde{a}_n x^n + \tilde{a}_{n-1} x^{n-1} + \cdots + \tilde{a}_1 x + p^k \in \mathbb{Z}[x],$$

gives rise to a canonical number system, then by [20, Theorem 5.3],

$$\left(\frac{\tilde{a}_n}{p^k}, \dots, \frac{\tilde{a}_1}{p^k} \right) \in \mathcal{D}_n^0.$$

Thus, the archimedean roots of

$$\tilde{f}(x) := x^n \tilde{g}\left(\frac{1}{x}\right) = p^k x^n + \tilde{a}_1 x^{n-1} + \cdots + \tilde{a}_{n-1} x + \tilde{a}_n$$

are strictly inside the complex unit circle. If we define

$$a_j := -\frac{\tilde{a}_j}{p^k}$$

for $j \in \{1, \dots, n\}$, then the polynomial

$$x^n - a_1x^{n-1} - \dots - a_n$$

fulfills (5.5b).

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¹INSTITUT FÜR MATHEMATIK, UNIVERSITÄT FÜR BODENKULTUR, GREGOR-MENDEL-STRASSE 33, A-1180 VIENNA, AUSTRIA
E-mail address: `klaus.scheicher@boku.ac.at`

²DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD SIMÓN BOLÍVAR, APARTADO 89000, CARACAS 1086-A, VENEZUELA
E-mail address: `vsirvent@usb.ve`

³ENGERTHSTRASSE 52, A-1200 VIENNA, AUSTRIA
E-mail address: `palovsky@yahoo.com`